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Vol. 59, No. 1, August 1985
Printed in Belgium

On the Nonlinear Diffusion Equation of Kolmogorov, Petrovsky, and Piscounov

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Received September 20, 1983; revised March 14, 1984

Recently, Bramson proved a theorem that classifies the initial data under which solutions of the K–P–P equation converge to the appropriate travelling waves. In this paper, a simplified proof is given by using maximal principles instead of his Brownian motion approach. The regularity condition on the forcing term is also weakened. © 1985 Academic Press, Inc.

1. INTRODUCTION

In this paper, we will consider the semilinear diffusion equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+F(u), \tag{1.1}$$

where the forcing term F is assumed to be in $C^1[0, 1]$ and satisfies the conditions

$$F(0)=F(1)=0, \qquad F(u)>0 \qquad \text{for} \quad 0<u<1$$

and

$$F'(0)=1, \qquad F'(u)\leqslant 1 \qquad \text{for} \quad 0<u\leqslant 1.$$

One of the main interests of this equation is the asymptotic behavior of the solution for large time. This problem was first investigated by Kolmogorov, Petrovsky, and Piscounov [7] under the Heaviside initial data, and was applied to study some population genetic models by Fisher [5]. We will call such an equation the K–P–P equation. During the past 2 decades,

* Supported by NSF Grants MCS-7903638, MCS-8203328, and a grant from the Institute for Advanced Study.

0022-0396/85 $3.00

there has been a considerable amount of work done on this subject (e.g., Aronson and Weiberger [1, 2], Bramson [3], Fife and McLeod [6], McKean [8] and Uchiyama [9]). For complete references and development of this subject, the readers may refer to Bramson [4].

Let $u(t, x)$ denote a solution of (1.1). By a travelling wave with speed λ we mean a function w^λ which satisfies

$$w^\lambda(x) = u(t, x + \lambda t), \qquad -\infty < x < \infty, \quad t \geqslant 0.$$

Clearly, w^λ satisfies the ordinary differential equation

$$\tfrac{1}{2} w^\lambda_{xx} + \lambda w^\lambda_x + F(w^\lambda) = 0.$$

By excluding trivial cases, we may assume that $w(-\infty) = 1$, and $w(\infty) = 0$. A simple phase plane analysis shows that for nontrivial $0 \leqslant w^\lambda \leqslant 1$ to exist, it is necessary that $\lambda \geqslant 2$ ([3, 4, 9]). Let $b = \lambda - \sqrt{\lambda^2 - 2}$; then for $\lambda > \sqrt{2}$, $w^\lambda(x) \sim ce^{-bx}$ for large x, and for $\lambda = \sqrt{2}$, $w^\lambda(x) \sim cxe^{-\sqrt{2}x}$ provided that $F'(u) = 1 - o(u^p)$ for some $0 < p < 1$. Recently, Bramson [3] proved the following theorem, which classifies the approach of initial data to appropriate travelling waves.

THEOREM. *Suppose F satisfies* (1.1) *and the additional condition $F'(u) = 1 - o(u^p)$ for some $0 < p < 1$. Then*

(i) *For $\lambda > \sqrt{2}$,*

$$u(t, x + m(t)) \to w^\lambda(x)$$

uniformly in x as $t \to \infty$ for some choice of $m(t)$ if and only if for some (all) $h > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \log \int_x^{(1+h)x} u(0, y)\, dy = -b \tag{1.2}$$

($b = \lambda - \sqrt{\lambda^2 - 2}$), *and for some $\eta > 0$, $M > 0$, $N > 0$,*

$$\int_x^{x+N} u(0, y)\, dy > \eta \qquad \textit{for} \quad x < -M. \tag{1.3}$$

(ii) *For the case $\lambda = \sqrt{2}$, condition* (1.2) *can be replaced by*

$$\lim_{x \to \infty} \frac{1}{x} \log \int_x^{(1+h)x} u(0, y)\, dy \leqslant -\sqrt{2}.$$

In particular, the sufficient conditions considered by Kolmogorov, Petrovsky, and Piscounov [7], and Uchiyama [9] are special cases of the theorem. Also, Bramson showed that for $\lambda > \sqrt{2}$ the above $m(t)$ can be chosen as $\sup\{x: \phi(t, x) \geqslant 1\}$, where

$$\phi(t, x) = e^t \int_{-\infty}^{\infty} u(0, y) \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} dy.$$

For $\lambda = \sqrt{2}$, and $\int_0^\infty xe^{\sqrt{2}x} u(0, x)\, dx < \infty$, $m(t)$ can be chosen as

$$m(t) = \sqrt{2}t - 3 \cdot 2^{-3/2} \log t;$$

for $\int_0^\infty xe^{\sqrt{2}x} u(0, x)\, dx = \infty$, the centering term $m(t)$ has been obtained for certain important classes of initial data. For example, if $h(x) = e^{\sqrt{2}x} u(0, x)$ has order $0(x^\gamma)$, $\gamma > 0$, then $m(t)$ can be chosen as

$$m(t) = \sqrt{2}t - 3 \cdot 2^{-3/2} \log t + b(t),$$

where $b(t) = (1/\sqrt{2}) \log[\int_0^\infty xh(x) e^{-x^2/2t}\, dt] \vee 0$.

Bramson's proof is mainly probabilistic, i.e., it depends heavily on the Feynman–Kac formula and some Brownian motion methods. The proof is rather lengthy, especially for the case $\lambda = \sqrt{2}$. In this paper, we will give a simpler analytic proof of the same theorem, with a slight relaxation of the hypotheses on F.

The main part of the proof is on the sufficiency. The estimation depends on an implicit expression of the solution (see (2.2)). The basic tool is the extended maximal principle, which has been used by McKean [8], Uchiyama [9] and Bramson [3]. Our poof is divided into two parts: For the case $\lambda > \sqrt{2}$, where the initial data satisfy (1.2), we will show that for some $\varepsilon > 0$, the solution behaves "nicely' on the curve $\{(0, x): x \leqslant 0\} \cup \{(\varepsilon x, x): x \geqslant 0\}$ so that Uchiyama's argument can be modified. For the case $\lambda = \sqrt{2}$, we will approximate the initial data f by

$$f_r(x) = (f(x) + e^{-\delta r} e^{-bx}) \wedge 1,$$

where $b = \sqrt{2} - 2\delta$ for some suitable $\delta > 0$. The new function f_r satisfies the smooth conditions of Uchiyama (f can be assumed differentiable). Moreover, if we let

$$m(t) = \sup\{x: u(t, x; f) \geqslant \tfrac{1}{2}\}$$

and let $m^r(t)$ be the corresponding function for f_r, then $m^r(t)$ will be close to $m(t)$ for a long period of time (say, $0 \leqslant t \leqslant r/4\delta$, Lemma 3.3), and then drift away for large values of t. Letting $t_r = r/4\delta$, we are then able to apply the extended maximal principle to compare $u(t_r, x + m(t_r); f)$ with $w^\lambda(x)$ ($\sqrt{2} - 4\delta = \lambda - \sqrt{\lambda^2 - 2}$) via f_r (i.e., compare the functions before $m^r(t)$ drifts away from $m(t)$).

Our paper is organized as follows: In Section 2 we give some preliminaries for the K–P–P equation. We prove the sufficiency for the case $\lambda = \sqrt{2}$ in Section 3. The proof of the case $\lambda > \sqrt{2}$ is in Section 4, whereas its lemmas are proved separately in Section 5. In Section 6 we prove the necessity of the theorem.

2. The K–P–P Equation

Let $p_t(x) = p(t, x)$ denote the normal density $\sqrt{2\pi t}^{-1} e^{-(x-y)^2/2t}$, let $0 \leqslant f \leqslant 1$ be initial data and let $u(t, x; f) = u(t, x)$ be the solution of the K–P–P equation (1.1). It follows from a simple application of the fixed point theorem that $u(t, x)$ satisfies

$$u(t, x) = p_t * f(x) + \int_0^t \int_{-\infty}^{\infty} p(t-s, x-y)\, F(u(s, y))\, dy\, ds, \tag{2.1}$$

and $0 \leqslant u(t, x) \leqslant 1$ for $-\infty < x < \infty$, $t \geqslant 0$. Let $F(u) = u - \theta(u)$, then $0 \leqslant \theta(u) \leqslant u$ and $\lim_{u \to 0} \theta(u)/u = 0$. By using a recursive argument, we obtain

$$u(t, x) = e^t p_t * f(x) - E(x, t), \tag{2.2}$$

where

$$E(x, t) = e^t \int_0^t e^{-s} \int_{-\infty}^{\infty} p(t-s, x-y)\, \theta(u(s, y))\, dy\, ds.$$

For the linear diffusion equation

$$\frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + v$$

with initial data f, the solution is $v(t, x) = e^t p_t * f(x)$. Obviously, $u(t, x) \leqslant v(t, x)$ for $-\infty < x < \infty$, $t \geqslant 0$.

Throughout, we will assume that f satisfies either

$$0 \leqslant f \leqslant 1, \qquad \lim_{x \to \infty} \frac{1}{x} \log \int_x^{(1+h)x} f(y)\, dy \leqslant -b \tag{2.3}$$

or

$$0 \leqslant f \leqslant 1, \qquad \lim_{x \to \infty} \frac{1}{x} \log \int_x^{(1+h)x} f(y)\, dy = -b \tag{2.4}$$

where $0<b\leqslant\sqrt{2}$, and h is some (and hence any) positive number. It is clear that (2.3) ((2.4)) is equivalent to

$$\int_x^{(1+h)x} f(y)\,dy \leqslant (=) e^{-b-o(1))x}. \tag{2.5}$$

LEMMA 2.1. *Suppose f satisfies* (2.3); *then for any* $0<t_0<t$, $0<b'<b$,

(i) $\lim_{x\to\infty} e^{b'x} p_s * f(x)=0$ *uniformly for* $t_0\leqslant s\leqslant t$,

(ii) $p_t * f$ *satisfies* (2.3) (*and satisfies* (2.4) *if f does*).

Proof. (i) Follows from the following inequalities: for $t_0\leqslant s\leqslant t$,

$$p_s * f(x)\leqslant \int_{(1-h)x}^{(1+h)x} p(s, x-y) f(y)\,dy + 2\int_{-\infty}^{(1-h)x} p(s, x-y)\,dy$$

$$\leqslant \frac{1}{\sqrt{2\pi t_0}} e^{-(b+o(1))x} + 2\sqrt{2} e^{-h^2x^2/4t}.$$

(ii) Inequality (2.3) also follows from the above inequality. To show that (2.4) holds, we need only make use of the following:

$$p_t * f(x) \geqslant \int_{x-1}^{x+1} p(t, x-y)\, d(y)\,dy \geqslant \frac{e^{-1/2t}}{\sqrt{2\pi t}} \int_{x-1}^{x+1} f(y)\,dy. \quad ∎$$

LEMMA 2.2. *Suppose f satisfies* (2.3), *then for any* $t>0$, $\lim_{x\to\infty} E(t, x)/e^t p_t * f(x)=0$.

Proof. For any $\varepsilon>0$, there exist δ, $x_0>0$ such that

(i) $0<\theta(u)=\varepsilon u$ for $u<\delta$,

(ii) $u(s, y)<\delta$ for $y\geqslant x_0$, $\varepsilon\leqslant s\leqslant t-\varepsilon$ (by Lemma 2.1),

(iii) $\int_{-\infty}^{x_0} p(t-s, x-y)\,dy \leqslant \varepsilon p_t * f(x)$, $x>2x_0$, $\varepsilon\leqslant s\leqslant t-\varepsilon$.

(Condition (iii) holds since for $x>2x_0$, $\varepsilon\leqslant s\leqslant t-\varepsilon$,

$$\int_{-\infty}^{x_0} p(t-s, x-y)\,dy \leqslant \sum_{k=0}^{\infty} \frac{e^{-(x-x_0+k)^2/2(t-s)}}{\sqrt{2\pi(t-s)}} = o(e^{-(x-x_0)^2/2t}),$$

whereas $p_t * f(x)\geqslant k e^{-x^2/2t}$ for some $k>0$). Hence for $x>2x_0$, $\varepsilon\leqslant s\leqslant t-\varepsilon$,

$$\int_{x_0}^{\infty} p(t-s, x-y)\,\theta(u(s, y))\,dy \leqslant \varepsilon \int_{x_0}^{\infty} p(t-s, x-y)\, u(s, y)\,dy \quad \text{(by (i), (ii))}$$

$$\leqslant \varepsilon e^s \int_{-\infty}^{\infty} p(t-s, x-y)\, p_s * f(y)\,dy$$

$$= \varepsilon e^s p_t * f(x).$$

This, together with (iii), yields

$$\int_{-\infty}^{\infty} p(t-s, x-y)\,\theta(u(s, y))\,dy \leqslant 2\varepsilon e^s p_t * f(x).$$

Therefore, for $x > 2x_0$,

$$0 \leqslant E(t, x) = e^t\left(\int_0^{\varepsilon} + \int_{\varepsilon}^{t-\varepsilon} + \int_{t-\varepsilon}^{t}\right) e^{-s}\int_{-\infty}^{\infty} p(t-s, x-y)\,\theta(u(s, y))\,dy\,ds$$

$$\leqslant 4\varepsilon t e^t p_t * f(x).$$

This implies $\lim_{x\to\infty} E(t, x)/e^t p_t * f(x) = 0$. ∎

PROPOSITION 2.3. *Suppose f satisfies* (2.3), ((2.4)), *then for any $t > 0$, $u(t, x)$ satisfies* (2.3) ((2.4), *respectively*), *and for any $0 < b' < b$,*

$$\lim_{x\to\infty} e^{b'x} u(t, x) = \lim_{x\to\infty} e^{b'x} \frac{\partial}{\partial x} u(t, x) = 0.$$

Proof. The first statement and the first limit follow from Lemmas 2.1, 2.2. To prove the second limit, we need only observe that

$$\lim_{x\to\infty} e^{b'x}\int_{-\infty}^{\infty}\left|\frac{x-y}{t}\right| p(t, x-y)\,f(y)\,dy = 0,$$

and

$$\lim_{x\to\infty} \frac{\partial E(t, x)}{\partial x}\Big/\int_{-\infty}^{\infty}\left|\frac{x-y}{t}\right| p(t, x-y)\,f(y)\,dy = 0.$$

(The proof is similar to Lemmas 2.1 and 2.2.) This implies that

$$\lim_{x\to\infty} e^{b'x}\frac{\partial u}{\partial x} = \lim_{x\to\infty} e^{b'x}\left(e^t \frac{\partial}{\partial x} p_t * f(x) - \frac{\partial}{\partial x} E(t, x)\right) = 0. \quad ∎$$

The following maximal principle holds for the K–P–P equation, it can be derived from [3, Proposition 3.1].

PROPOSITION 2.4. *Let g, h be initial data of the* K–P–P *equation and $0 \leqslant g \leqslant h \leqslant 1$, then*

$$0 \leqslant u(t, x; h) - u(t, x; g) \leqslant v(t, x; h-g), \quad -\infty < x < \infty, \qquad t \geqslant 0.$$

Let g, h be two measurable functions on R; according to the notation of

Bramson [3, 4], we say that g is *more stretched* than h, denoted by $g \overset{s}{\geqslant} h$ or $h \overset{s}{\leqslant} g$), if for any $c>0$, and $x_1 \leqslant x_2$,

$$g(x_1) \geqslant h(x_1+c) \Rightarrow g(x_2) \geqslant h(x_2+c).$$

Intuitively, $g \overset{s}{\geqslant} h$ says that h falls faster than $g(x)$ as x increases. If g and h are differentiable, then $g \overset{s}{\geqslant} h$ means that g lies above h in its mutual phase plane. The following theorem was proved by McKean [8] by using the Feynman–Kac formula and Brownian motion (see also [3, Proposition 3.2]). It can also be proved by analytic methods ([9, Proposition 3.4]).

THEOREM 2.5 (Extended maximal principle). *Let $0 \leqslant g, h \leqslant 1$ be initial data of the K–P–P equation, and assume that $g \overset{s}{\geqslant} h$. Then $u(t, x; g) \overset{s}{\geqslant} u(t, x; h)$. Moreover,*

$$\begin{aligned} u(t, x+m^g(t); g) &\geqslant u(t, x+m^h(t); h), \qquad x \geqslant 0 \\ &\leqslant u(t, x+m^h(t); h), \qquad x<0. \end{aligned}$$

where $m^g(t)=\sup\{x: u(t, x; g) \geqslant \frac{1}{2}\}$.

In Section 5, we will need a more general form of the extended maximal principle. For any $\varepsilon>0$, let ζ_ε denote the curve

$$\begin{aligned} \zeta_\varepsilon(x) &= \varepsilon x \qquad \text{if} \quad x \geqslant 0 \\ &= 0 \qquad \text{if} \quad x \leqslant 0. \end{aligned}$$

THEOREM 2.6. *Let $0 \leqslant g, h \leqslant 1$ be initial data of the K–P–P equation. Suppose there exists an $\varepsilon>0$ such that*

$$u(\zeta_\varepsilon(x), x; g) \overset{s}{\geqslant} u(\zeta_\varepsilon(x), x; h),$$

then for each $t>0$,

$$u(t, x; g) \overset{s}{\geqslant} u(t, x; h), \qquad -\infty<x \leqslant \varepsilon^{-1} t.$$

The proof is the same as Theorem 2.5 with some obvious adjustment, both in analytic and probabilistic approaches.

We will conclude this section with two more propositions ([3, Lemma 3.3, and Corollary 1 of Lemma 3.2]).

PROPOSITION 2.7. *There exists a constant c such that for any* $0 \leqslant g, h \leqslant 1$ *with* $g(x) = h(x)$ *for* $x \geqslant x_0$, *then*

$$|u(t, x; g) - u(t, x; h)| \leqslant ct^{-1/4} \tag{2.6}$$

for $x > x_0 + \sqrt{2}t - 2^{-5/2} \log t$.

For $\lambda > \sqrt{2}$ *if both* $u(t, x + m^g(t); g)$, $u(t, x + m^h(t); h) \to w^\lambda(x)$ *uniformly as* $t \to \infty$, *and satisfy* (2.6), *then it is easy to show* $m^g(t) - m^h(t) \to 0$ *as* $t \to \infty$.

PROPOSITION 2.8. *Let f be initial data satisfying* (2.3). *Suppose* $u(t, x + m(t)) \to w^\lambda(x)$ *uniformly in x with* $t \to \infty$ *for some* $m(t)$ *as* $\lambda \geqslant \sqrt{2}$; *then* $m(t)/t \to \lambda$ *as* $t \to \infty$.

3. THE CASE $\lambda = \sqrt{2}$

In this section, we will prove the following theorem:

THEOREM 3.1. *Let the K–P–P equation and F be defined as in* (1.1). *Let* $0 \leqslant f \leqslant 1$ *be initial data and satisfy* (i)

$$\lim_{x \to \infty} \frac{1}{x} \log \int_x^{(1+h)x} f(y)\, dy \leqslant -\sqrt{2} \tag{3.1}$$

for some $h > 0$, (ii) *for some* $\eta > 0$, $M > 0$, $N > 0$,

$$\int_x^{x+N} f(y)\, dy > \eta \qquad \text{for} \quad x < -M \tag{3.2}$$

Then $u(t, x + m(t)) \to w^{\sqrt{2}}(x)$ *uniformly on x as* $t \to \infty$.

Let H be the Heaviside function and let f be defined as above, then $H \lessgtr f$, and by Theorem 2.5,

$$\begin{aligned} u(t, x + m(t); f) &\geqslant u(t, x + m^H(t); H), \qquad x \geqslant 0 \\ &\leqslant u(t, x + m^H(t); H), \qquad x < 0. \end{aligned}$$

It is well known that $u(t, x + m^H(t); H)$ converges to $w^{\sqrt{2}}(x)$ uniformly on x as $t \to \infty$. Hence for any $\varepsilon > 0$ and for t large,

$$\begin{aligned} u(t, x + m(t); f) &\geqslant w^{\sqrt{2}}(x) - \varepsilon, \qquad x \geqslant 0 \\ &\leqslant w^{\sqrt{2}}(x) + \varepsilon, \qquad x < 0. \end{aligned} \tag{3.3}$$

The main effort is to prove the reverse inequality, i.e., for any $\varepsilon > 0$ and t large,

$$\begin{aligned} u(t, x+m(t); f) &\leqslant w^{\sqrt{2}}(x) + \varepsilon, \qquad x \geqslant 0 \\ &\geqslant w^{\sqrt{2}}(x) - \varepsilon, \qquad b(t) \leqslant x < 0, \end{aligned} \tag{3.4}$$

where $b(t) \to -\infty$ as $t \to \infty$. Thus (3.3), (3.4) imply that $u(t, x+m(t))$ converges uniformly on $[-N, \infty)$ as $t \to \infty$, for any positive integer N. Condition (3.2) will enable us to conclude that the convergence is actually uniform on $(-\infty, \infty)$.

By Proposition 2.3, we may assume without loss of generality that f is differentiable and for any $\delta > 0$, $f(x), f'(x) \leqslant e^{-(\sqrt{2}-\delta)x}$ for large x. We define

$$f_r(x) = (f(x) + e^{-\delta r} e^{-(\sqrt{2}-2\delta)x}) \wedge 1.$$

LEMMA 3.2. *There exists $r_\delta > 0$ such that for $x > r > r_\delta$,*

$$f_r'(x) \leqslant -(\sqrt{2} - 3\delta) f_r(x).$$

Proof. Let r_0 be such that for $x > r_0$,

$$f(x), f'(x) \leqslant e^{-(\sqrt{2}-\delta)x}.$$

Let $c = \delta^{-1}(\log(1 + \sqrt{2} - 3\delta) - \log \delta)$, and let $r_\delta = r_0 + c$, then it is easy to show that for $x > r > r_\delta$,

$$\begin{aligned} f_r'(x) &= f'(x) - (\sqrt{2} - 2\delta) e^{-\delta r} e^{-(\sqrt{2}-2\delta)x} \\ &\leqslant -(\sqrt{2} - 3\delta) f_r(x). \quad \blacksquare \end{aligned}$$

LEMMA 3.3. *Let $\delta > 0$ be fixed, and let $r/4\delta \leqslant t \leqslant (1+3\delta)r/4\delta$, then for $x \geqslant (\sqrt{2} - \frac{5}{4}\delta^2)t$,*

$$0 \leqslant u(t, x; f_r) - u(t, x; f) \leqslant \eta_1(r),$$

where $\eta_1(r) \to 0$ as $r \to \infty$.

Proof. By using the maximal principle (Proposition 2.4), we have

$$0 \leqslant u(t, x; f_r) - u(t, x; f) \leqslant v(t, x; f_r - f),$$

where $v(t, x; g) = e^t p_t * g(x)$. It follows that for $x \geqslant (\sqrt{2} - \frac{5}{4}\delta^2)t$,

$$\begin{aligned} v(t, x; (f_r - f)) &\leqslant e^t \int_{-\infty}^{\infty} p_t(x-y) e^{-\delta r} e^{-(\sqrt{2}-2\delta) y} \, dy \\ &= e^{t - \delta r - (\sqrt{2}-2\delta)x + (\sqrt{2}-2\delta)^2 t/2} \\ &\leqslant e^{-\delta r + (\frac{5}{4}\sqrt{2}+2)\delta r/4 + o(\delta^2) r}. \end{aligned}$$

The proof is completed by letting the last expression equal $\eta_1(r)$. ∎

Let $b = \sqrt{2} - 4\delta$, $\lambda = 1/b + b/2$ (i.e., $b = \lambda - \sqrt{\lambda^2 - 2}$). Let $w^\lambda(x)$ be the travelling wave with speed λ and normalized to $w^\lambda(0) = \frac{1}{2}$, then $\lim_{x \to \infty} w^\lambda(x)/e^{-bx} = c$, for some $c > 0$. For each $r > r_\delta$ we define $\bar{r}$ so that $w^\lambda(\bar{r}) = f_r(r)$, then there exists $r_1 > r_\delta$ such that for $r > r_1$,

$$\bar{r} \geqslant r \qquad \text{and} \qquad 2ce^{-b\bar{r}} \geqslant w^\lambda(\bar{r}) = f_r(r) \geqslant e^{-(\sqrt{2}-\delta)r}.$$

(The first inequality holds since $w^\lambda(r) \geqslant f_r(r)$ for large r, and w_λ is decreasing.) This implies that

$$\frac{|\log 2c|}{\sqrt{2}-4\delta} + \left(1 + \frac{3\delta}{\sqrt{2}-4\delta}\right) r \geqslant \bar{r}.$$

Furthermore if r_1 is chosen greater than $|\log 2c|/3\delta(\sqrt{2}-4\delta-1)$, then $r \leqslant \bar{r} \leqslant (1+3\delta)r$. We define

$$\begin{aligned} \bar{w}^\lambda(x) &= w^\lambda(x), \qquad & x \geqslant \bar{r} \\ &= 0, & x < \bar{r}. \end{aligned}$$

The following lemma is analogous to Proposition 2.7, but different in that $\bar{w}^\lambda$ and t change according to r.

LEMMA 3.4. *Let $t = \bar{r}/4\delta$, then for $x > (\lambda - \frac{5}{4}\delta^2)t$,*

$$0 \leqslant u(t, x; w^\lambda) - u(t, x; \bar{w}^\lambda) \leqslant \eta_2(r),$$

where $\eta_2(r) \to 0$ as $r \to \infty$.

Proof. In view of Proposition 2.4 and the proof of last lemma, we need only make the following estimation: for $x > (\lambda - \frac{5}{4}\delta^2)t$,

$$v(t, x; w^\lambda - \bar{w}^\lambda) \leqslant \frac{2ce^t}{\sqrt{2\pi t}} \int_{-\infty}^{\bar{r}} e^{-(x-y)^2/2t} e^{-by}\, dy$$

$$\leqslant \frac{2c}{\sqrt{2\pi t}} e^{t - bx + b^2 t/2} \int_{-\infty}^{0} e^{-(x - bt - \bar{r} - y)^2/2t}\, dy$$

$$\leqslant \frac{2c}{\sqrt{2\pi t}} e^{t - bx + b^2 t/2} e^{-(x - bt - \bar{r})^2/2t}$$

$$\leqslant \frac{2c}{\sqrt{2\pi t}} e^{5b\delta^2 t/4} e^{-(2\delta - \frac{5}{4}\delta^2)^2 t/2}.$$

Let $\eta_2(r)$ be the last expression. Since $r < 4\delta t < (1 + 3\delta) r$, it follows that $\eta_2(r) \to 0$ as $r \to \infty$. ∎

Let $m(s) = \sup\{x : u(s, x; \bar{w}^\lambda) \geqslant \frac{1}{2}\}$.

LEMMA 3.5. *Let $t = \bar{r}/4\delta$. Then*

$$0 \leqslant |w^\lambda(x) - u(t, x + \bar{m}(r); \bar{w}^\lambda)| \leqslant \eta_3(r), \qquad -\tfrac{1}{4}\delta^2 t < x < \infty,$$

where $\eta_3(r) \to 0$ as $r \to \infty$.

Proof. It follows from Lemma 3.4, the strict monotonicity and the continuity of w^λ that for large $\bar{r}$, we have $(\lambda - \delta^2) t \leqslant \bar{m}(t) \leqslant \lambda t$. Moreover,

$$\begin{aligned} 0 &\leqslant w^\lambda(\bar{m}(t) - \lambda t) - w^\lambda(0) \\ &= u(t, \bar{m}(t); w^\lambda) - \tfrac{1}{2} \\ &= u(t, \bar{m}(t); w^\lambda) - u(t, \bar{m}(t); \bar{w}^\lambda) \\ &\leqslant \eta_2(r). \end{aligned}$$

Again, by the strictly monotonicity and continuity of w^λ at 0, we have $\lim_{r \to \infty} (\bar{m}(t) - \lambda t) = 0$. Now by the uniform continuity of w^λ and by Lemma 3.4, we have for $-\frac{1}{4}\delta^2 t < x < \infty$,

$$\begin{aligned} &|w^\lambda(x) - u(t, x + \bar{m}(t); w^\lambda)| \\ &\qquad \leqslant |w^\lambda(x) - w^\lambda(x + \bar{m}(t) - \lambda t)| + |w^\lambda(x + \bar{m}(t) - \lambda t) - u(t, x + \bar{m}(t); w^\lambda)| \\ &\qquad \leqslant \varepsilon(r) + \eta_2(r), \end{aligned}$$

where $\varepsilon(r)$, $\eta_2(r) \to 0$ as $r \to \infty$. The proof is completed by letting $\eta_3(r) = \varepsilon(r) + \eta_2(r)$. ∎

Proof of Theorem 3.1. According to the remark in the beginning of this section, we will show that (3.4) holds. Let $m(s)$, $m^r(s)$, $m^H(s)$, $\bar{m}(s)$ correspond to the initial data f, f_r, H and $\bar{w}^\lambda$, respectively. Then for any $s>0$,

$$m^r(s) \geqslant m(s) \geqslant m^H(s) \geqslant (\sqrt{2}-\delta^2)s.$$

From the construction of f_r and $\bar{w}^\lambda$, we have $f_r \lesssim \bar{w}^\lambda$. Let $t=\bar{r}/4\delta$, then Theorem 2.5 and Lemmas 3.3 and 3.5 imply

$$\begin{aligned} u(t, x+m^r(r); f) &\leqslant w^\lambda(x)+\eta_1(r)+\eta_3(r), \qquad && x \geqslant 0 \\ &\geqslant w^\lambda(x)-\eta_1(r)-\eta_3(r), && -\frac{\delta^2}{4}t<x<0. \end{aligned} \tag{3.5}$$

It remains to replace $m^r(t)$ by $m(t)$. For this we let

$$a(r)=|u(t, m(t); f_r)-u(t, m^r(r); f_r)| = |u(t, m(t); f_r)-\tfrac{1}{2}|.$$

Then Lemma 3.3 implies that for r large and for $m(t)>(\sqrt{2}-\delta^2)t$, $a(r) \leqslant \eta_1(r)$. For each r let y be the real number closest to $\bar{m}(t)$, so that $y \leqslant \bar{m}(t)$ and

$$|u(t, y; \bar{w}^\lambda)-u(t, \bar{m}(t); \bar{w}^\lambda)| = |u(t, y; \bar{w}^\lambda)-\tfrac{1}{2}| = a(r).$$

Since $f_r \lesssim \bar{w}^\lambda$, we have $0 \leqslant m^r(t)-m(t) \leqslant \bar{m}(t)-y \leqslant \lambda t-y$. Also

$$\begin{aligned} |w^\lambda(y-\lambda t)-w^\lambda(0)| &\leqslant |u(t, y; \bar{w}_\lambda)-\tfrac{1}{2}|+\eta_2(r) \\ &= a(r)+\eta_2(r) \\ &\leqslant \eta_1(r)+\eta_2(r). \end{aligned}$$

This implies that $\lim_{r\to\infty}(y-\lambda t)=0$. Therefore, we have $m^r(t)\to m(t)$ as $t\to\infty$ and

$$u(t, x+m(t); f) = u(t, x+m(t)-m^r(t)+m^r(t); f)$$
$$\begin{cases} \leqslant w^\lambda(x)+(\eta_1(\bar{r})+\eta_3(r)+\eta_4(r)), & x \geqslant 0 \\ \geqslant w^\lambda(x)-(\eta_1(\bar{r})+\eta_3(r)+\eta_4(r)), & -\dfrac{\delta^2}{4}t \leqslant x \leqslant 0 \end{cases}$$

where $\eta_4(r)\to 0$ as $r\to\infty$.

To conclude the proof, we still have to show that the convergence is uniform on $(-\infty, \infty)$. It suffices to prove that

$$u(t, x+m(t); f) \to w^{\sqrt{2}}(x) \text{ uniformly for } x \leqslant -\frac{\delta^2}{4}t \text{ as } t\to\infty.$$

We may assume, without loss of generality, that $\alpha = \inf\{f(y): y \leqslant 0) > 0$ (for otherwise we will use $u(t_0, y)$, $t_0 > 0$, instead of f as initial data). Let

$$\bar{H}(x) = 0, \qquad x \geqslant 0$$
$$= \alpha, \qquad x < 0.$$

By applying the part we just proved, we see that $u(t, x + m^{\bar{H}}(t); \bar{H}) \to w^{\sqrt{2}}(x)$ uniformly on $[-N, \infty)$, for any $N > 0$, as $t \to \infty$. Since $\bar{H}$ is decreasing, a simple application of the maximal principle will imply that $u(t, x + m^{\bar{H}}(t); \bar{H}) \to w^{\sqrt{2}}(x)$ uniformly on $(-\infty, \infty)$ as $t \to \infty$. In particular, for any $\varepsilon > 0$, there exists t_0 and $N > 0$ such that

$$1 \geqslant u(t, x + m^{\bar{H}}(t); \bar{H}) \geqslant 1 - \varepsilon \qquad \text{for} \quad t \geqslant t_0, \ x < -N. \tag{3.6}$$

Note that $\lim_{t \to \infty} m^{\bar{H}}(t)/t = \sqrt{2}$ (Proposition 2.8), and $\overline{\lim}_{t \to \infty} m(t)/t < \lambda$ for $\lambda > \sqrt{2}$ (this can be proved directly by showing that $\lim_{t \to \infty} u(t, \lambda t) \leqslant \lim_{t \to \infty} e^t p_t * f(\lambda t) = 0$, $\lambda > \sqrt{2}$). Hence $-(\delta^2/4)t + (m(t) - m^{\bar{H}}(t)) \to -\infty$ as $t \to \infty$. For $x < -(\delta^2/4)t$, we have

$$1 \geqslant u(t, x + m(t); f) \geqslant u(t, x + m(t) - m^{\bar{H}}(t)) + m^{\bar{H}}(t); \bar{H}).$$

This combined with (3.6) implies the claim. ∎

4. The Case $0 < b < \sqrt{2}$

The main theorem of this section is

THEOREM 4.1. *Let F be as in (1.1) and satisfy $F(u) = u - \theta(u)$, where $\theta(u) = o(u \log^{-\rho} u)$ for some $\rho > 1$. Let $0 \leqslant f \leqslant 1$ be initial data of the K–P–P equation and satisfy*: (i)

$$\lim_{x \to \infty} \frac{1}{x} \log \int_x^{(1+h)x} f(y)\, dy = -b \tag{4.1}$$

for some $0 < b < \sqrt{2}$ and $h > 0$; (ii) *there exists η, M, $N > 0$ such that*

$$\int_x^{x+N} f(y)\, dy > \eta \qquad \text{for} \quad x < -M. \tag{4.2}$$

Then $u(t, x + m(t)) \to w^{\lambda}(x)$ uniformly on x as $t \to \infty$.

The idea of proof is to show that f behaves nicely on the curve

$$\{(0, x) : x \leqslant 0\} \cup \{(\varepsilon x, x) : x \geqslant 0\}$$

so that the technique in [9] is applicable by using the modified extended maximal principle (Theorem 2.6).

LEMMA 4.2. *Suppose f satisfies* (4.1), *then for any $\eta > 0$, there exists ε_0 and $x_\varepsilon > 0$, $0 < \varepsilon < \varepsilon_0$, such that for $x > x_\varepsilon$,*

$$-(b+\eta)\, u(\varepsilon x, x) \leqslant u'(\varepsilon x, x) \leqslant -(b-\eta)\, u(\varepsilon x, x).$$

LEMMA 4.3. *Let f, η and ε_0 be defined as in the last lemma. Then for $0 < \varepsilon < \varepsilon_0$, there exists an x_1 (depending on ε) such that for $\bar{x} \geqslant x_1$,*

$$-(b+2\eta)\, u(\varepsilon x, x; \bar{f}) \leqslant u'(\varepsilon x, x; \bar{f}), \qquad x \geqslant 0,$$

where

$$\begin{aligned} \bar{f}(x) &= f(x), & x &\geqslant \bar{x} \\ &= 0, & x &< \bar{x}. \end{aligned}$$

The proofs of these two lemmas are conceptually simple, but deviate from the main proof of this section, therefore we will postpone them to next section.

Proof of Theorem 4.1. *Step* 1. We claim that for any $\delta > 0$, there exists a t_0 such that for $t > t_0$,

$$\begin{aligned} u(t, x+m(t); f) &\geqslant w^\lambda(x) - \delta, & 0 &\leqslant x \leqslant \varepsilon^{-1} t \\ &\leqslant w^\lambda(x) + \delta, & (\sqrt{2}-\lambda)t &\leqslant x < 0. \end{aligned} \tag{4.3}$$

Let $0 < b < \bar{b} < \sqrt{2}$, $\eta = \bar{b} - b$ and $\bar{\lambda} = 1/\bar{b} + \bar{b}/2$. Since $w^{\bar{\lambda}}$ is a travelling wave,

$$u(\varepsilon x, x; w^{\bar{\lambda}}) = w^{\bar{\lambda}}((1-\bar{\lambda}\varepsilon)x).$$

Let ε_0 be defined as in Lemma 4.2. For $0 < \varepsilon < \varepsilon_0$, there exists x_0 such that for $x > x_0$,

$$(w^{\bar{\lambda}}((1-\bar{\lambda}\varepsilon)x))' \leqslant -\left(\bar{b} - \frac{n}{2}\right) w^{\bar{\lambda}}((1-\bar{\lambda}\varepsilon)x).$$

We claim that there is a large $\bar{x}$ so that for $x \geqslant 0$:

(i) $u(\varepsilon x, x; \bar{f}) \leqslant w^{\bar{\lambda}}((1-\bar{\lambda}\varepsilon)x_0)$;

(ii) $-(b+n/2)\, u(\varepsilon x, x; \bar{f}) \leqslant u'(\varepsilon x, x; \bar{f})$.

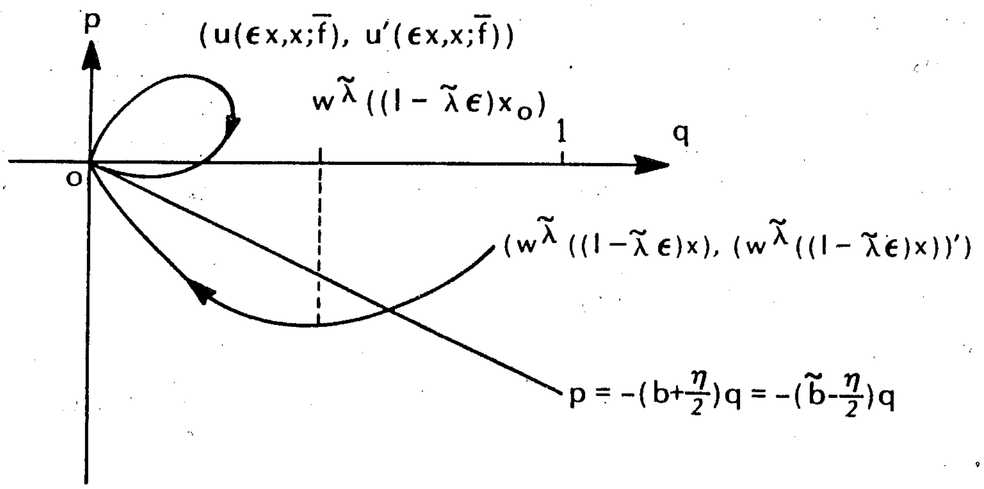

FIGURE 1

Indeed, a direct estimation of

$$u(\varepsilon x, x; \bar{f}) = \int_{\bar{x}}^{\infty} p(\varepsilon x, x - y) f(y)\, dy$$

shows that $u(\varepsilon x, x; f)$ is increasing on $[0, \bar{x} - \varepsilon]$. This combined with $u(\varepsilon x, x; \bar{f}) \leqslant u(\varepsilon x, x; f)$ for all x, $u(\varepsilon x, x; f) \leqslant w^{\tilde{\lambda}}((1 - \tilde{\lambda}\varepsilon)x)$ for large x, and $w^{\tilde{\lambda}}$is decreasing imply (i). Part (ii) follows from Lemma 4.3.

Let $\zeta_\varepsilon(x) = \varepsilon x$ if $x \geqslant 0$, 0 if $x \leqslant 0$. If follows from the above construction that $u(\zeta_\varepsilon(x), x; \bar{f}) \gtrless w^{\tilde{\lambda}}((1 - \tilde{\lambda}\zeta_\varepsilon)x)$ (see the definition of stretchness and the phase plane diagram above). Theorem 2.6 implies that

$$\begin{aligned} u(t, x + \bar{m}(t); \bar{f}) &\geqslant w^{\tilde{\lambda}}(x), \qquad 0 \leqslant x \leqslant \varepsilon^{-1} t \\ &\leqslant w^{\tilde{\lambda}}(x), \qquad x < 0, \end{aligned}$$

and hence we have, from Proposition 2.7, that

$$\begin{aligned} u(t, x + \bar{m}(t); f) &\geqslant w^{\tilde{\lambda}}(x) - \eta(t), \qquad 0 \leqslant x \leqslant \varepsilon^{-1} t \\ &\leqslant w^{\tilde{\lambda}}(x) + \eta(t), \qquad h(t) \leqslant x \leqslant 0, \end{aligned}$$

where $\lim_{t \to \infty} h(t) = -\infty$. The claim now follows by replacing $\bar{m}(t)$ by $m(t)$, and by observing that $w^{\tilde{\lambda}} \to w^{\lambda}$ uniformly on x as $\tilde{\lambda} \to \lambda$.

Step 2. We will prove the reversed inequalities as in Step 1, i.e., for any $\delta > 0$, there exists t_1 such that for $t > t_1$,

$$\begin{aligned} u(t, x + m(t); f) &\leqslant w^{\lambda}(x) + \delta, \qquad 0 \leqslant x \leqslant \varepsilon^{-1} t \\ &\geqslant w^{\lambda}(x) - \delta, \qquad (\sqrt{2} - \lambda)t \leqslant x < 0. \end{aligned} \tag{4.4}$$

By assumption (4.2) we may assume that $\inf_{y \leqslant 0} f(y) > 0$ and $f > 0$ (otherwise, we consider $u(t_0, x)$ for some $t_0 > 0$). Since $\lim_{x \to \infty} u(\varepsilon x, x; f) = 0$, for any $y > 0$, we can find $z > y$ so that $u(\varepsilon x, x; f) \geqslant u(\varepsilon z, z; f)$ for $0 < x < z$. Let $0 < \hat{b} < b < \sqrt{2}$ and let $\eta = b - \hat{b}$. By applying Lemma 4.2 to f and Lemma 4.3 to w^{λ}, we can find $\varepsilon > 0$, x_0, $\bar{x}$ such that

(i) $u'(\varepsilon x, x; f) \leqslant -(b - \eta/4)\, u(\varepsilon x, x; f)$ for $x \geqslant x_0$,

(ii) $-(b + \eta/4)\, u(\varepsilon x, x; \bar{w}^{\lambda}) \leqslant (\varepsilon x, x; \bar{w}^{\lambda})$ for $x \geqslant 0$,

(iii) $u(\varepsilon x, x; f) \geqslant u(\varepsilon x_0, x_0; f)(= a)$ for $0 \leqslant x \leqslant x_0$, $u(\varepsilon x, x, \bar{w}^{\lambda}) \leqslant a$ for $x \geqslant 0$

(the second part of (iii) can be done by choosing $\bar{x}$ large enough and apply similar argument as in Step 1(ii)).

It follows that $u(\varepsilon x, x; f) \overset{s}{\leqslant} u(\varepsilon x, x; \bar{w}_{\lambda})$ (see Fig. 2). By Theorem 2.6,

$$
\begin{aligned}
u(t, x + m(t); f) &\leqslant u(t, x + \bar{m}(t); \bar{w}^{\lambda}), && 0 \leqslant x \leqslant \varepsilon^{-1} t \\
&\geqslant u(t, x + \bar{m}(t); \bar{w}^{\lambda}), && x < 0,
\end{aligned}
$$

and by Proposition 2.7

$$
\begin{aligned}
u(t, x + m(t); f) &\leqslant w^{\lambda}(x) + \eta(t), && 0 \leqslant x \leqslant \varepsilon^{-1} t \\
&\geqslant w^{\lambda}(x) - \eta(t), && (\sqrt{2} - \hat{\lambda}) t \leqslant x \leqslant 0,
\end{aligned}
$$

where $\eta(t) \to 0$ as $t \to \infty$. Hence (4.4) follows by observing that $w^{\lambda} \to w^{\lambda}(x)$ uniformly on $-\infty < x < \infty$ as $t \to \infty$.

Step 3. The previous argument implies that $u(t, x + m(t); f)$ converges uniformly on bounded intervals as $t \to \infty$. We claim that the convergence is

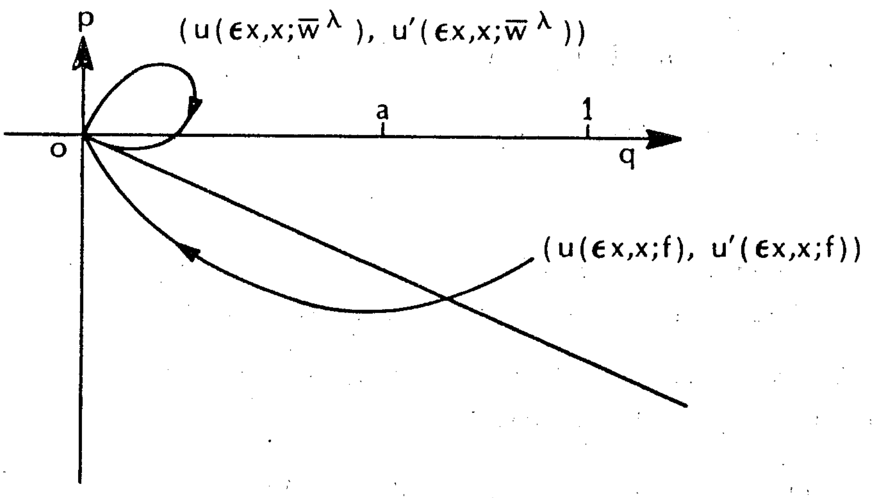

FIGURE 2

uniform on $(-\infty, N)$, where N is a positive integer. By hypothesis (4.2), we may assume that $\inf_{y\leqslant x} f(y)>0$. For any $0<b<b'<\sqrt{2}$, there exists an x' such that the function

$$g(x)=e^{-b'x}, \qquad x\geqslant x'$$
$$=e^{-b'x'}, \qquad x<x',$$

satisfies $u(\zeta_\varepsilon(x), x; g)\leqslant u(\zeta_\varepsilon(x), x: f)$ for all $-\infty<x<\infty$. The claim follows by applying the same argument as the last part of the proof of Theorem 3.1, with $\bar{H}$ replaced by g.

The proof of the uniform convergence of $u(t, x+m(t); f)$ on (N, ∞) is similar. We assume without loss of generality that f satisfies the condition in Lemma 2.1. Let

$$h(x)=w^{\lambda''(x)}, \qquad x\geqslant x''$$
$$=1, \qquad x<x'',$$

where $\sqrt{2}<\lambda<\lambda''$, and $f(x)\leqslant h(x)$. The uniform convergence of $u(t, x+m(t); h)$ to zero on $[e^{-1}t, \infty)$ implies the same for $u(t, x+m(t); f)$. ∎

5. Proof of the Lemmas

We will prove Lemmas 4.2 and 4.3 in this section. Throughout we assume that f satisfies (4.1). For any $0<\alpha, \delta, \varepsilon<1$ satisfying $0<\delta<1$, $\varepsilon(b+\delta)<1$, $4\sqrt{b\varepsilon}<1$, $\sqrt{\varepsilon}((b+\delta)+4\sqrt{\delta})<4\sqrt{b}$. Let

$$\bar{x}=(1-\varepsilon(b+\delta))x, \qquad h=\frac{2\varepsilon\delta}{1-\varepsilon(b+\delta)}, \qquad n_\delta=\left[\frac{2}{\sqrt{\delta}}\right].$$

It is obvious that

$$1+h=\frac{1-\varepsilon(b-\delta)}{1-\varepsilon(b+\delta)}, \qquad (1+h)\bar{x}=(1-\varepsilon(b-\delta))x, \qquad x-\bar{x}=\varepsilon(b+\delta)x.$$

Also

$$(1+h)^{-n_\delta}\bar{x}\geqslant\left(1-\frac{2}{\sqrt{\delta}}\right)\bar{x}=(1-\varepsilon(b+\delta)-4\varepsilon\sqrt{\delta})x\geqslant(1-4\sqrt{b\varepsilon})x.$$

Let E_i, $i=0, 1, 2, 3$, be a subdivision of the real line defined by

$$E_0=\{y: (1+h)^{-n_\delta}\bar{x}<y\leqslant(1+h)\bar{x}\},$$
$$E_1=\{y: |x-y|\geqslant 4\sqrt{b\varepsilon x}\},$$
$$E_2=\{y: (1-4\sqrt{b\varepsilon})x=y\leqslant(1+h)^{-n_\delta}\bar{x}\}$$

and

$$E_3 = \{y : (1+h)\bar{x} < y < (1+4\sqrt{b\varepsilon})x\}.$$

We will use the notation

$$\phi_\varepsilon(x, E_i) = \int_{E_i} p(\varepsilon x, x-y) f(y)\, dy.$$

In the following lemma we show that for small ε, the convolution $p_{\varepsilon x} * f(x)$ is concentrated in E_0 when x is large.

LEMMA 5.1. *There exists ε_0, δ_0 such that for $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$,*

$$\lim_{x\to\infty} \phi_\varepsilon(x, E_i) \Big/ \int_{\bar{x}}^{(1+h)\bar{x}} p(\varepsilon x, x-y) f(y)\, dy = 0, \qquad i = 1, 2, 3. \quad (5.1)$$

Proof. We will consider the three cases separately:

(i) For $i = 1$, we have

$$0 \leqslant \phi_\varepsilon(x, E_1) \leqslant \frac{1}{\sqrt{2\pi\varepsilon x}} \int_{|x-y| > 4\sqrt{b\varepsilon x}} e^{-(x-y)^2/2\varepsilon x}\, dx$$
$$\leqslant \frac{1}{\sqrt{2\pi\varepsilon x}} e^{-8bx}.$$

On the other hand,

$$\phi_\varepsilon(x, E_0) > \int_{\bar{x}}^{(1+h)\bar{x}} p(\varepsilon x, x-y) f(y)\, dy$$
$$\geqslant p(\varepsilon x, x-\bar{x}) \int_{\bar{x}}^{(1+h)\bar{x}} f(y)\, dy$$
$$\geqslant \frac{1}{\sqrt{2\pi\varepsilon x}} e^{-(b+((b+\delta)^2/2)\varepsilon)x + o(x)}.$$

The two inequalities imply (5.1).

(ii) For $i = 2$, we observe that if $(1+h)^{-n}\bar{x} \in E_2$, then

$$(1+h)^{-n} \geqslant \frac{(1-4\sqrt{b\varepsilon})x}{\bar{x}} \geqslant \frac{1-4\sqrt{b\varepsilon}}{1-\varepsilon(b+\delta)}.$$

Let n_1 denote the largest n so that $(1+h)^{-n}\bar{x} \in E_2$. For $n_\delta < n < n_1$, we define

$$Q_n(x) = \frac{\phi_\varepsilon(x, F_n)}{\phi_\varepsilon(x, F)},$$

where $F = [\bar{x}, (1+h)\bar{x}]$, $F_n = [(1+h)^{-n-1}\bar{x}, (1+h)^{-n}\bar{x}]$ for $n_\delta \leqslant n < n_1$, and $F_{n_1} = [(1-4\sqrt{b\varepsilon})x, (1+h)^{-n_1}\bar{x}]$. Then for $n_\delta \leqslant n \leqslant n_1$,

$$Q_n(x) = \frac{p(\varepsilon x, x-(1+h)^{-n}\bar{x})\, e^{-b((1+h)^{-n-1}\bar{x}+o_n(\bar{x}))}}{p(\varepsilon x, x-\bar{x})\, e^{-b\bar{x}+o(\bar{x})}}$$
$$= e^{a_n\bar{x}+o_n(\bar{x})-o(\bar{x})}$$

where

$$a_n = b(1-(1+h)^{-n-1}) - (1-(1+h)^{-n})\left(b+\delta+\frac{\delta}{h}(1-(1+h)^{-n})\right)$$

and

$$o_n(\bar{x}) = o((1+h)^{-n-1}\bar{x}).$$

We claim that $a_n < 0$. Indeed,

$$a_n = (1+h)^{-n-1}bh - (1-(1+h)^{-n})\left(\delta+\frac{\delta}{h}(1-(1+h)^{-n})\right)$$
$$< bh - (1-(1+h)^{-n})^2\frac{\delta}{h}$$
$$< bh - \frac{1}{2\varepsilon}(1-(1+h)^{-n}\delta)^2$$
$$= bh - \frac{1}{2\varepsilon}(n_\delta h + o(n_\delta h))^2$$
$$= (2b\delta\varepsilon + o(\delta\varepsilon)) - (8\delta\varepsilon + o(\delta\varepsilon)).$$

Hence for δ, ε sufficiently small, we have $a_n < 0$. This implies that $\lim_{x\to\infty} Q_n(x) = 0$ for $n_\delta \leqslant n < n_1$ and (5.1) follows.

(iii) The case $i=3$ involves estimation of the terms

$$\phi_\varepsilon(x, [(1+h)^n\bar{x}, (1+h)^{n+1}\bar{x}])/\phi_\varepsilon(x, F).$$

The technique in (ii) applies similarly.

COROLLARY 5.2. *There exist* ε_0, δ_0 *such that for* $0<\varepsilon<\varepsilon_0$, $0<\delta<\delta_0$,

$$\lim_{x\to\infty}\frac{\int_{(1+h)^{-n_\delta}\bar{x}}^{(1+h)\bar{x}}(x-y)^j\,p(\varepsilon x, x-y)\,f(y)\,dy}{\int_{-\infty}^{\infty}(x-y)^j\,p(\varepsilon x, x-y)\,f(y)\,dy}=1, \qquad j=0, 1, 2. \quad (5.2)$$

Proof. The case for $j=0$ is a direct consequence of Lemma 5.1. For the case $j=1, 2$, we need only apply the same technique to show that

$$\lim_{x\to\infty}\frac{\int_{E_i}(x-y)^j\,p(\varepsilon x, x-y)\,f(y)\,dy}{\int_{E_0}(x-y)^j\,p(\varepsilon x, x-y)\,f(y)\,dy}=0$$

for $i=1, 2, 3$, $j=1, 2$, as in Lemma 5.1. ∎

Recall that the solution of

$$\frac{\partial v}{\partial t}=\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+v$$

with initial condition f is $v(t, x)=e^t p_t * f(x)$.

PROPOSITION 5.3. *For any* $\eta>0$, *there exists* ε_0 *such that for* $0<\varepsilon<\varepsilon_0$, *we can find an* x_ε *which satisfies*

$$-(b+\eta)\,v(\varepsilon x, x)\leqslant v'(\varepsilon x, x)\leqslant -(b-\eta)\,v(\varepsilon x, x), \qquad x\geqslant x_\varepsilon. \quad (5.3)$$

Proof. It follows from direct calculation that $v'(\varepsilon x, x)=I_0-I_1+I_2$, where

$$I_0=\left(\varepsilon-\frac{1}{2x}\right)e^{\varepsilon x}\int_{-\infty}^{\infty}p(\varepsilon x, x-y)\,f(y)\,dy=\left(\varepsilon-\frac{1}{2x}\right)v(\varepsilon x, x)$$

$$I_1=e^{\varepsilon x}\int_{-\infty}^{\infty}\frac{(x-y)}{\varepsilon x}\,p(\varepsilon x, x-y)\,f(y)\,dy$$

and

$$I_2=e^{\varepsilon x}\int_{-\infty}^{\infty}\frac{(x-y)^2}{2\varepsilon x^2}\,p(\varepsilon x, x-y)\,f(y)\,dy.$$

Let ε and δ be as in Lemma 5.1; then for $i=1, 2$,

$$\int_{(1+h)^{-n_\delta}\bar{x}}^{(1+h)\bar{x}}(x-y)^i\,p(\varepsilon x, x-y)\,f(y)\,dy$$

$$\leqslant (x-(1+h)^{-n_\delta}\bar{x})^i\int_{(1+h)^{-n_\delta}\bar{x}}^{(1+h)\bar{x}}p(\varepsilon x, x-y)\,f(y)\,dy$$

$$\leqslant (x-(1-4\sqrt{\delta\varepsilon}+o(\sqrt{\delta\varepsilon}(\bar{x}))^i(1+o(x)\,p_{\varepsilon x}*f(x)$$

$$=x^i(b\varepsilon+o(\sqrt{\delta}\,\varepsilon))^i(1+o(x))\,p_{\varepsilon x}*f(x),$$

and

$$\int_{(1+h)^{-n_\delta}\bar{x}}^{(1+h)\bar{x}} (x-y)^i p(\varepsilon x, x-y) f(y)\,dy$$

$$\geqslant (x-(1+h)\bar{x})^i(1+o(x))\, p_{\varepsilon x} * f(x) \qquad \text{(by (5.2))}$$

$$= x^i(b\varepsilon + o(\delta\varepsilon))^i(1+o(x))\, p_{\varepsilon x} * f(x).$$

If we choose ε_0, δ_0 small enough, the above estimation and Lemma 5.2 implies inequality (5.3). ∎

LEMMA 5.4. *For any $\eta > 0$, there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, there exists a z_0 (which depends on ε) such that for $z > z_0$*

$$-(b+\eta)\, v(\varepsilon x, x; f_z) \leqslant v'(\varepsilon x, x; f_z), \qquad x \geqslant 0, \tag{5.4}$$

where $f_z(x) = 0$ if $x \leqslant 2$, and $f_z(x) = f(x)$ if $x \geqslant z$.

Proof. Let $\bar{x}$, h be defined as before. For any $\eta > 0$, there exists an ε_0 such that for each $0 < \varepsilon < \varepsilon_0$, there exists an x_ε and for $x > x_\varepsilon$,

$$\int_{-\infty}^{\infty} p(\varepsilon x, x-y) f(y)\,dy \leqslant \left(1 + \frac{n}{2}\right) \int_{(1+h)^{-n_\delta}\bar{x}}^{(1+h)\bar{x}} p(\varepsilon x, x-y) f(y)\,dy \tag{5.5}$$

and

$$-\left(b + \frac{\eta}{2}\right) v(\varepsilon x, x; f) \leqslant v'(\varepsilon x, x; f). \tag{5.6}$$

Let $z_0 = (1+h)^{-n_\delta}\bar{x}_\varepsilon = (1+h)^{-n_\delta}(1-\varepsilon(b+\delta))x_\varepsilon$, and for $z > z_0$ let $x_z = (1+h)^{n_\delta}(1-\varepsilon(b+\delta))^{-1}z$. We will show that (5.4) holds on the following three sets:

(i) If $0 \leqslant x \leqslant (1+\varepsilon^2)z$, then

$$v'(\varepsilon x, x; f_z) = e^{\varepsilon x} \int_z^{\infty} \left(\varepsilon - \frac{1}{2x} - \frac{x-y}{\varepsilon x} + \frac{(x-y)^2}{2\varepsilon x^2}\right) p(\varepsilon x, x-y) f(y)\,dy$$

$$= e^{\varepsilon x} \int_z^{\infty} \left(\frac{1}{2\varepsilon}\left(\frac{x-y}{x} - 1\right)^2 - \left(\frac{1}{2\varepsilon} - \varepsilon + \frac{1}{2x}\right)\right) p(\varepsilon x, x-y) f(y)\,dy$$

$$\geqslant 0.$$

This implies (5.1).

(ii) For $(1+\varepsilon^2)z \leqslant x \leqslant x_z$, we will perform the following estimates:

$$-\int_z^\infty \frac{(x-y)}{\varepsilon x} p(\varepsilon x, x-y) f(y)\,dy \geqslant -\int_z^x \frac{(x-y)}{\varepsilon x} p(\varepsilon x, x-y) f(y)\,dy$$

$$\geqslant -\frac{(x_z-z)}{\varepsilon x}\int_z^\infty p(\varepsilon x, x-y) f(y)\,dy$$

$$= -(b+o(\sqrt{\delta}))\int_z^\infty p(\varepsilon x, x-y) f(y)\,dy.$$

Hence

$$v'(\varepsilon x, x; f_z) \geqslant \left(\varepsilon - \frac{1}{2x} - b + o(\sqrt{\delta})\right)\cdot e^{\varepsilon x}\int_z^\infty p(\varepsilon x, x-y) f(y)\,dy,$$

and (5.4) holds if the preassigned ε_0, δ_0 are small and x_z is large.

(iii) For $x_z \leqslant x$, (5.5) and (5.6) will imply (5.4). ∎

In Section 2, we showed that the solution of the K–P–P equation is of the form $u(t, x) = v(t, x) - E(t, x)$, and $\lim_{x\to\infty} E(t, x)/v(t, x) = 0$ if the initial data f satisfy (2.1). In the following, we will show that $E(\varepsilon x, x)$, $E'(\varepsilon x, x)$ are also small compared to $v(\varepsilon x, x)$ as $x \to \infty$. We will omit the proof of the following lemma which is similar to Lemma 2.1 (we need only split the integral into $(-\infty, x)$ and (x, ∞) for large x).

LEMMA 5.5. *Suppose $0 < \varepsilon < 1/b$, then for any $0 < b' < b$, $\lim_{x\to\infty} e^{b'x} p_t * f(x) = 0$ uniformly in the region $\{(t, x)\colon t_0 \leqslant t \leqslant \varepsilon x\}$ for $t_0 > 0$.*

PROPOSITION 5.6. *Suppose $F(u) = \theta(u)$, where $\theta(u) = o(u \log^{-\rho} u)$ for some $\rho > 1$. Then there exists ε_0 such that for $0 < \varepsilon < \varepsilon_0$,*

(a) $\lim_{x\to\infty} E(\varepsilon x, x)/v(\varepsilon x, x) = 0$

(b) $\lim_{x\to\infty} E'(\varepsilon x, x)/v'(\varepsilon x, x) = \lim_{x\to\infty} E'(\varepsilon x, x)/v(\varepsilon x, x) = 0.$

Proof. For any $\eta > 0$, there exists δ_0, ε_0, $x_0 > 0$ such that

(i) $0 < \theta(u) < \eta\, |u \log^{-\rho} u|$ for $u < \delta_0$,

(ii) $\max\{u(s, y) \colon x/2 \leqslant y\} \leqslant e^{-bx/4}$ for $\eta < s < \varepsilon_0 x$, $x \geqslant x_0$,

(iii) $\int_{-\infty}^{x/2} p(\varepsilon x - s, x - y)\,dy \leqslant \eta p_{\varepsilon x} * f(x)$ for $\eta < s < \varepsilon_0 x$, $x \geqslant x_0$.

(Part (ii) follows from Lemma 5.5; (iii) follows from the same argument as in the first part of the proof in Lemma 5.1.) For $x \geqslant 2x_0$, $\eta \leqslant s \leqslant \varepsilon_0 x$,

$$\int_{x/2}^{\infty} p(\varepsilon x-s,\, x-y)\, \theta(u(s,\, y))\, dy$$

$$\leqslant \eta \int_{x/2}^{\infty} p(\varepsilon x-s,\, x-y)\cdot u(s,\, y)\cdot |\log^{-\rho} u(s,\, y)|\, dy$$

$$\leqslant \eta \max\left\{|\log^{-\rho} u(s,\, y)|\colon \frac{x}{2}\leqslant y\right\}\cdot \int_{x/2}^{\infty} p(\varepsilon x-s,\, x-y)\, e^{s} p_{s} * f(y)\, dy$$

$$\leqslant \eta \left(\frac{bx}{4}\right)^{-\rho} e^{s} p_{\varepsilon x} * f(x).$$

Now, by the same argument as Lemma 2.2, we can show that for $x > x_0$,

$$0 \leqslant E(\varepsilon x,\, x) \leqslant k\eta e^{\varepsilon x} p_{\varepsilon x} * f(x)$$

for some $k > 0$. This completes the proof of (a).

To prove (b), we note that

$$E'(\varepsilon x,\, x) = \varepsilon E(\varepsilon x,\, x) + 0(u(\varepsilon x,\, x))$$
$$+ e^{\varepsilon x} \int_{0}^{\varepsilon x} e^{-s} \int_{-\infty}^{\infty} p'(\varepsilon x,\, x-y)\, \theta(u(s,\, y))\, dy\, ds, \qquad (5.7)$$

and

$$p'(\varepsilon x-s,\, x-y) = \left(\frac{x-y}{\varepsilon x-s}\right) p(\varepsilon x-s,\, x-y)$$
$$+ \varepsilon\left(\frac{(x-y)^2}{2(\varepsilon x-s)^2} + \frac{1}{2(\varepsilon x-s)}\right) p(\varepsilon x-s,\, x-y)$$
$$= \frac{\partial}{\partial x} p(\varepsilon x-s,\, x-y) + \frac{\varepsilon}{2} \frac{\partial^2}{\partial x^2} p(\varepsilon x-s,\, x-y),$$

where $\partial/\partial x$ denotes the derivative on the second variable. We will estimate the last term of (5.7). First, we claim that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} \left|\frac{x-y}{\varepsilon x-s}\right| p(\varepsilon x-s,\, x-y)\, p_s * f(y)\, dy \Big/ -\left(\frac{\partial}{\partial x} p_{\varepsilon x}\right) * f(x) = 1. \qquad (5.8)$$

Indeed, Corollary 5.2 implies that the integral $\int_{-\infty}^{\infty} (\partial/\partial x)\, p(\varepsilon x,$ $x-y)\, f(y)\, dy$ is concentrated at $((1+h)^{-n_\delta}\bar{x},\ (1+h)\bar{x}) \subseteq (-\infty,\, x)$ for large value of x. This combined with

$$\int_{-\infty}^{\infty} \left(\frac{x-y}{\varepsilon x-s}\right) p(\varepsilon x,\, x-y)\, p_s * f(y)\, dy = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} p(\varepsilon x,\, x-y)\, f(y)\, dy \qquad (5.9)$$

implies (5.8). Now by using the analogous reasoning as in the first part, we have, for suitable ε_0, δ_0, x_0 and for $0<\varepsilon<\varepsilon_0$, $x>x_0$,

$$0 \leqslant -e^{\varepsilon x}\int_0^{\varepsilon x} e^{-s}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}p(\varepsilon x-s,x-y)\,\theta(u(s,y))\,dy\,ds$$

$$\leqslant e^{\varepsilon x}\int_0^{\varepsilon x} e^{-s}\int_{-\infty}^{\infty} e^{-s}\int_{-\infty}^{\infty}\left|\frac{x-y}{\varepsilon x-s}\right| p(\varepsilon x-s,x-y)\,\theta(u(s,y))\,dy\,ds$$

$$\leqslant -\eta(1+o(1))e^{\varepsilon x}\left(\frac{\partial}{\partial x}p_{\varepsilon x}\right)*f(x).$$

where $o(1)\geqslant 0$ and $\lim_{x\to\infty} o(1)=0$. Again by the same argument, we can adjust the above ε_0, x_0, δ_0 so that for $0<\varepsilon<\varepsilon_0$, $x>x_0$,

$$0\leqslant e^{\varepsilon x}\int_0^{\varepsilon x}\cdot e^{-s}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}p(\varepsilon x-s,x-y)\,\theta(u(s,y))\,dy\,ds$$

$$\leqslant \eta e^{\varepsilon x}\left(\frac{\partial^2}{\partial x^2}p_{\varepsilon x}\right)*f(x).$$

(Note that in this case, $(\partial^2/\partial x^2)\,p(\varepsilon x-s,\ x-y)$ is positive, so the complication involving the absolute value in (5.8) will not appear). Hence for the above ε and x.

$$\left|e^{\varepsilon x}\int_0^{\varepsilon x} e^{-s}\int_{-\infty}^{\infty} p'(\varepsilon x-s,x-y)\,\theta(u(s,y))\,dy\,ds\right|$$

$$\leqslant -\eta(1+o(1))e^{\varepsilon x}\left(\frac{\partial}{\partial x}p_{\varepsilon x}\right)*f(x)+\eta\frac{\varepsilon}{2}e^{\varepsilon x}\left(\frac{\partial^2}{\partial x^2}p_{\varepsilon x}\right)*f(x).$$

Now part (b) follows from (5.7), Proposition 5.3, the estimation of I_1, I_2 in Proposition 5.3 and the above inequality. ∎

Proof of Lemma 4.2. It follows directly from Proposition 5.3 and Proposition 5.6. ∎

Proof of Lemma 4.3. Let $\eta>0$, let $\bar{x}$ and $\bar{f}$ be define as in Lemma 4.2. If $\bar{x}$ is chosen large enough, then parts (ii) and (iii) in the proof of Proposition 5.6 can be modified to

(ii)′ $\max\{u(s,y;\bar{f})\colon x/2<y\}\leqslant e^{-bx/4}$, $\eta<s<\varepsilon_0 x$, $x\geqslant 0$,

(iii)′ $\int_{\bar{x}}^{\max\{\bar{x},x/2\}} p(\varepsilon x,x-y)\,dy\leqslant \eta p_{\varepsilon x}*\bar{f}(x)$,
$|\int_{\bar{x}}^{\max\{\bar{x},x/2\}} p'(\varepsilon x,x-y)\,dy|\ \leqslant\ \eta\,|p'_{\varepsilon x}*\bar{f}(x)|$, $\eta<s<\varepsilon_0 x$, $x\geqslant\bar{x}$,

(iv)′ $\int_{-\infty}^{\infty}|(x-y)/(\varepsilon x-s)|\,p(\varepsilon x-s,x-y)\,p_s*\bar{f}(y)\,dy$
$\leqslant -(1+\eta)((\partial/\partial x)\,p_{\varepsilon x})*\bar{f}(x)$, $x\geqslant 0$.

(Part (iv)$'$ can be obtained by a modification of the proof of (5.8).) By using this, Proposition 5.6 can be strengthened to: there exists ε_0 such that for any $0<\varepsilon<\varepsilon_0$ and for any $\bar{\eta}>0$, there exists $\bar{x}$ with

$$E(\varepsilon x, x; \bar{f}) \leqslant \bar{\eta} v(\varepsilon x, x), \ |E'(\varepsilon x, x; \bar{f})| \leqslant \bar{\eta}\, |v'(\varepsilon x, x)|, \qquad x \geqslant 0.$$

This and Lemma 5.4 will imply Lemma 4.3. ∎

6. The Necessity

We will prove the converse of Theorem 3.1 and Theorem 4.1 together, the extra condition of F in Theorem 4.1 is not needed.

In [3, Proposition 4.2], Bramson showed that condition (3.1) or (4.1) holds if and only if $\lim_{t\to\infty}(1/t)\log v(t, \lambda t)=0$, i.e., $v(t, \lambda t)=e^{o(t)}$.

Assuming that $\lim_{t\to\infty} u(t, m(t)+x)=w^{\lambda}(x)$ uniformly on x, we want to show that $\lim_{t\to\infty}(1/t)\log v(t, \lambda t)=0$. That $\underline{\lim}_{t\to\infty}(1/t)\log v(t, \lambda t)\geqslant 0$ follows from the first part of proof of [3, Theorem 2] with no change. We will make a slight change on the proof of $\overline{\lim}_{t\to\infty}(1/t)\log v(t, \lambda t)\leqslant 0$ in order to free it from the Brownian motion argument.

Supposing this is not true, we can find a $\delta>0$ such that

$$\overline{\lim_{t\to\infty}}\, e^{-\delta t} v(t, (\lambda+\delta)t)>1 \tag{6.1}$$

[3, Lemma 4.4]. Since $u(t, m(t)+x)\to w^{\lambda}(x)$ uniformly for $x\in(-\infty, \infty)$ as $t\to\infty$, we have $\lim_{t\to\infty} m(t)/t=\lambda$. As $\lim_{x\to\infty} w^{\lambda}(x)=0$, for any $0<\eta<\delta$ there exists t_1 such that

$$\theta(u(s, y))<\eta u(s, y) \text{ on } \left\{(s, y): \left(\lambda+\frac{\delta}{2}\right)s\leqslant y,\ t_1\leqslant s\right\}.$$

It follows from (2.1) that

$$\begin{aligned} u(t, x) &= p_t * f(x)+\int_0^t\int_{-\infty}^{\infty} p(t-s, x-y)\, F(u(s, y))\, dy\, ds \\ &\geqslant p_t * f(x)+(1-\eta)\int_{t_1}^t\int_{(\lambda+\delta/2)s}^{\infty} p(t-s, x-y)\, u(s, y)\, dy\, ds \\ &\geqslant p_t * f(x)+(1-\eta)\int_{t_1}^t\int_{-\infty}^{\infty} p(t-s, x-y)\, u(s, y)\, dy\, ds-\alpha_1 \end{aligned} \tag{6.2}$$

$$\text{where } \alpha_1 = (1-\eta)\int_{t_1}^{t}\int_{-\infty}^{(\lambda+\delta/2)s} p(t-s, x-y)\, dy\, ds$$

$$= (1+(1-\eta)(t-t_1))\, p_t * f(x) + (1-\eta)\int_{t_1}^{t}(t-s)\int_{-\infty}^{\infty} p(t-s, x-y)$$

$$\cdot F(u(s, y))\, dy\, ds - \alpha_1 \qquad (\text{by } (6.2))$$

$$= \sum_{n=0}^{\infty} \frac{(1-\eta)^n (t-t_1)^n}{n!}\, p_t * f(x) + \lim_{n\to\infty} (1-\eta)^n \int_{t_1}^{t} \frac{(t-s)^n}{n!}$$

$$\cdot \int_{-\infty}^{\infty} p(t-s, x-y)\, F(u(s, y))\, dy\, ds - \sum_{n=1}^{\infty} \alpha_n$$

$$\text{where } \alpha_n = (1-\eta)^n \int_{t_1}^{t} \frac{(t-s)^n}{n!} \int_{-\infty}^{(\lambda+\delta/2)s} p(t-s, x)-y)\, dy\, ds$$

$$= e^{(1-\eta)(t-t_1)} p_t * f(x) - \sum_{n=1}^{\infty} \alpha_n$$

$$= v(t, x) \cdot e^{(1-\eta)(t-t_1)-t} - \sum_{n=1}^{\infty} \alpha_n.$$

Note that

$$\sum_{n=1}^{\infty} \alpha_n = \int_{t_1}^{t} e^{(1-\eta)(t-s)} \int_{-\infty}^{(\lambda+\delta/2)s} p(t-s, x-y)\, dy\, ds$$

and for $x = (\lambda+\delta)t$, the integral can be shown to tend to zero as t tends to infinity. By (6.1), and $0 < \eta < \delta$, we have

$$\varlimsup_{t\to\infty} u(t, (\lambda+\delta)t) \geqslant \varlimsup_{t\to\infty} v(t, (\lambda+\delta)t)\, e^{(1-\eta)(t-t_1)-t}$$

$$\geqslant \varlimsup_{t\to\infty} e^{\delta t} \cdot e^{(1-\eta)(t-t_1)-t} = \infty.$$

This contradicts $0 \leqslant u(t, x) \leqslant 1$.

Finally, the condition (3.2) (and (4.2)) follows from the same argument of Bramson [3, Sect. 5, Theorem 2].

Acknowledgments

The author would like to express his graditude to Professor S. T. Yau for bringing his attention to this problem, and to Professor M. Bramson for making this preprints available. He also wishes to thank the referee for many valuable comments in preparing the paper.

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium